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On Walsh Equiconvergence

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INTRODUCTION

The theorem of J. L. Walsh [3, p. 153], which is the inspiration of our work, has the following setting. Let $A(\rho)$ $(1 < \rho < \infty)$ be the set of functions $f(z)$, analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then we put

$$S_n(f; z) = \sum_{j=0}^{n} a_j z^j, \tag{1}$$

and denote by $L_n(f; z)$ the polynomial of degree at most n which interpolates to f at the $(n+1)$-st roots of unity, i.e., if $\omega^{n+1} = 1$ then

$$L_n(f; \omega) = f(\omega).$$

The theorem of Walsh can then be stated as

THEOREM W. *If $f \in A(\rho)$, then*

$$\lim_{n\to\infty} (L_n(f; z) - S_n(f; z)) = 0, \qquad |z| < \rho^2, \tag{2}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \rho^2$. Moreover, the result is best possible, in the sense that (2) fails for every z satisfying $|z| = \rho^2$ for an $f \in A(\rho)$.

We wish to present some variations on this theme of Walsh equiconvergence. In Section 1, we generalize Theorem W by considering least-squares approximations to f by polynomials of degree n on the mth roots of unity $(m \geqslant n + 1)$ and proving analogues of Theorem W. The case $m = n + 1$ is just Walsh's result. Other generalizations of Walsh's theorem may be found in Cavaretta *et al.* [1]. In the second and final section, we consider

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Walsh-type equiconvergence among the functions analytic inside ellipses with foci at ± 1.

1

If $f \in A(\rho)$ and

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

then for $j = 0, 1, 2, \ldots$

$$a_j = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{j+1}} \, dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \, e^{-ij\varphi} \, d\varphi.$$

Let m be a positive integer. Put

$$\omega_{k,m} = e^{2\pi k i/m}, \qquad k = 0, \ldots, m-1.$$

Then it is easy to see that for every integer l

$$\begin{aligned} \frac{1}{m} \sum_{k=0}^{m-1} \omega_{k,m}^{l} &= 0, \qquad m \nmid l \\ &= 1, \qquad m \mid l. \end{aligned} \tag{3}$$

Consider the trapezoidal rule for a 2π-periodic function, $g(\varphi)$,

$$\frac{1}{2\pi} \int_0^{2\pi} g(\varphi) \, d\varphi = \frac{1}{m} \sum_{k=0}^{m-1} g\left(\frac{2\pi k}{m}\right) + R_m(g) =: T_m(g) + R_m(g). \tag{4}$$

Put

$$g_j(\varphi) = f(e^{i\varphi}) \, e^{-ij\varphi}, \qquad j = 0, 1, 2, \ldots,$$

then

$$\begin{aligned} a_j^{(m)} := T_m(g_j) &= \frac{1}{m} \sum_{k=0}^{m-1} f(\omega_{k,m}) \, \omega_{k,m}^{-j} = \frac{1}{m} \sum_{k=0}^{m-1} \omega_{k,m}^{-j} \left(\sum_{p=0}^{\infty} a_p \omega_{k,m}^{p} \right) \\ &= \sum_{p=0}^{\infty} a_p \left[\frac{1}{m} \sum_{k=0}^{m-1} \omega_{k,m}^{p-j} \right], \end{aligned} \tag{5}$$

and, in view of (3), we have

$$a_j^{(m)} = a_j + \sum_{s=1}^{\infty} a_{j+sm}, \qquad j = 0, 1, 2, \ldots, \tag{6}$$

a formula for the error made in approximating the power series coefficients by the trapezoidal rule.

Next, fix a positive integer n and suppose $m \geqslant n+1$. Put

$$p_{n,m}(f; z) = \sum_{j=0}^{n} \alpha_j^{(m)} z^j.$$

Then we claim that

$$p_{n,m}(f; z) = S_n(L_{m-1}(f); z), \tag{7}$$

where L_{m-1} is the polynomial of degree at most $m-1$ interpolating to f at $\omega_{k,m}$, $k = 0,..., m-1$. To establish (7) we need only observe that $\alpha_j^{(m)}$ remains unchanged if $f(\omega_{k,m})$ is replaced by $L_{m-1}(f; \omega_{k,m})$ in (5) and thus (6) implies that

$$L_{m-1}(f; z) = \sum_{j=0}^{m-1} \alpha_j^{(m)} z^j.$$

Furthermore, $p_{n,m}(f; z)$ is the least-squares approximation of degree n to f on $\{\omega_{0,m},..., \omega_{m-1,m}\}$. To see this, note that taking (7) into account

$$\begin{aligned} f(\omega_{k,m}) - p_{n,m}(f; \omega_{k,m}) &= f(\omega_{k,m}) - L_{m-1}(f; \omega_{k,m}) + \sum_{j=n+1}^{m-1} \alpha_j^{(m)} \omega_{k,m}^j \\ &= \sum_{j=n+1}^{m-1} \alpha_j^{(m)} \omega_{k,m}^j \end{aligned}$$

(the sum on the right-hand side being set equal to zero when $m = n+1$), and so the inner products

$$\sum_{k=0}^{m-1} [f(\omega_{k,m}) - p_{n,m}(f; \omega_{k,m})]\, \bar{\omega}_{k,m}^q = \sum_{j=n+1}^{m-1} \alpha_j^{(m)} \sum_{k=0}^{m-1} \omega_{k,m}^{j-q} = 0,$$

for $q = 0,..., n$ in view of (3).

Our main result is

THEOREM 1. *If $f \in A(\rho)$ and q is a fixed positive integer then*: (i)

$$\lim_{n\to\infty} (p_{n,m(n)}(f; z) - S_n(f; z)) = 0, \qquad |z| < \rho^{1+q}, \tag{8}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \rho^{1+q}$, where $m(n) := nq + c$, and c is a fixed integer. (ii) *Moreover, the result is best possible, in the sense that* (8) *fails for every z satisfying $|z| = \rho^{1+q}$ for an $f \in A(\rho)$.*

Proof. Given $f \in A(\rho)$ we have for any given n, in view of (6),

$$p_{n,m}(f; z) - S_n(f; z) = \sum_{j=0}^{n} (a_j^{(m)} - a_j)\, z^j = \sum_{j=0}^{n} \left(\sum_{s=1}^{\infty} a_{j+sm} \right) z^j. \tag{9}$$

Now, $f \in A(\rho)$ implies that

$$\overline{\lim_{k \to \infty}} |a_k|^{1/k} = \frac{1}{\rho};$$

hence, upon putting $(0 <)\, \varepsilon < (\rho - 1)/2$, there exists a $k_0(\varepsilon)$ such that for $k > k_0(\varepsilon)$,

$$|a_k| < \left(\frac{1}{\rho} (1 + \varepsilon) \right)^k. \tag{10}$$

Put

$$r = \frac{1 + \varepsilon}{\rho} < 1.$$

Suppose that $\tau \geqslant \rho$, then for $|z| \leqslant \tau$ and $n > k_0(\varepsilon)$, (9) and (10) yield

$$|p_{n,m}(f; z) - S_n(f; z)| \leqslant \sum_{j=0}^{n} \left(\sum_{s=1}^{\infty} r^{j+sm} \right) \tau^j = \frac{r^m}{1 - r^m} \frac{\tau^{n+1} r^{n+1} - 1}{\tau r - 1}. \tag{11}$$

If k_0 is now chosen so large that $r^n < \frac{1}{2}$ for $n > k_0$, (11) implies that

$$|p_{n,m}(f; z) - S_n(f; z)| \leqslant r^{m+n} \tau^n \left(\frac{2\tau r}{\tau r - 1} \right). \tag{12}$$

Hence, if we insist that in addition to $\tau > r^{-1}$ also

$$\tau < \frac{1}{r^{1+q}}, \tag{13}$$

then

$$|p_{n,m(n)}(f; z) - S_n(f; z)| \leqslant \left(\frac{2\tau r^c}{\tau r - 1} \right) (\tau r^{1+q})^n \tag{14}$$

for $n > k_0$. Since $\varepsilon > 0$ may be chosen as small as we wish, part (i) of the theorem follows from (13).

As for part (ii), we need only consider $f(z) = (1 - z/\rho)^{-1}$ and carry out a straightforward computation beginning with (9) to see that (8) fails for this function (the same one Walsh used to prove his result best possible) for all z satisfying $|z| = \rho^{1+q}$.

Remarks

1. Since $p_{n,m}$ is the least-squares approximation of degree n to f on the mth roots of unity, Walsh's theorem is the case $m = n + 1$ of Theorem 1.

2. Equation (7) implies that

$$p_{n,n+2}(f; z) = L_{n+1}(f; z) - \alpha_{n+1}^{(n+2)} z^{n+1}.$$

Thus

$$f(\omega_{k,n+2}) - p_{n,n+2}(f; \omega_{k,n+2}) = \alpha_{n+1}^{(n+2)} \omega_{k,n+2}^{n+1} = \alpha_{n+1}^{(n+2)} \bar{\omega}_{k,n+2},$$

from which we can conclude (cf. Rivlin [2]) that $p_{n,n+2}(f; z)$ is the best uniform approximation to f by a polynomial of degree at most n on the set $\{\omega_{0,n+2},..., \omega_{n+1,n+2}\}$ (with error $|\alpha_{n+1}^{(n+2)}|$), since

$$\sum_{k=0}^{n+1} \omega_{k,n+2} \omega_{k,n+2}^{j} = 0, \qquad j = 0,..., n.$$

Theorem 1 in this case is Theorem 7 of Cavaretta *et al.* [1].

3. As $q \to \infty$, $p_{n,m(n)}$ tends to S_n, which is the least-squares approximation to f on all of $|z| = 1$.

2

Suppose $1 < \rho < \infty$. Let C_ρ be the ellipse, in the z-plane, which is the image of the circle $|w| = \rho$, in the w-plane, under the mapping

$$z = \frac{w + 1/w}{2}.$$

Let $A(C_\rho)$ denote the set of functions, $f(z)$, analytic inside C_ρ and having a singularity on C_ρ. Let

$$f(z) = \sum_{k=0}^{\infty}{}' A_k T_k(z) \tag{15}$$

($T_k(z)$ is the Chebyshev polynomial of degree k and the stroke on a summation sign means that the first term of the sum is to be halved), where

$$A_k = \frac{2}{\pi} \int_{-1}^{1} f(x)\, T_k(x) \frac{dx}{\sqrt{1 - x^2}}, \qquad k = 0, 1,.... \tag{16}$$

Put

$$s_n(f; z) = \sum_{k=0}^{n}{}' A_k T_k(z), \qquad n = 0, 1, \dots. \tag{17}$$

We wish to consider analogues of Theorem W in which various sequences of polynomials will exhibit geometric equiconvergence with (17) in ellipses C_τ with $\tau > \rho$.

As analogues of the interpolation in the roots of unity, interpolation in the zeros or the extrema of the Chebyshev polynomials come to mind. Let us put

$$\xi_j^{(m)} = \cos \frac{(2j-1)\pi}{2m}, \qquad j = 1, \dots, m$$

and

$$\eta_j^{(m)} = \cos \frac{j\pi}{m}, \qquad j = 0, \dots, m,$$

so that

$$T_m(\xi_j^{(m)}) = 0, \qquad j = 1, \dots, m$$

and

$$T_m(\eta_j^{(m)}) = (-1)^j, \qquad j = 0, \dots, m.$$

A straightforward computation will verify that

$$\sum_{j=1}^{m} T_\mu(\xi_j^{(m)})\, T_\nu(\xi_j^{(m)}) = \begin{cases} \dfrac{(-1)^p + (-1)^q}{2} m, & \left\{ \begin{array}{l} \mu + \nu = 2pm, \\ \text{and} \\ |\mu - \nu| = 2qm \end{array} \right. \\ (-1)^s \dfrac{m}{2}, & \left\{ \begin{array}{l} \mu + \nu = 2sm \\ \text{and} \\ |\mu - \nu| \neq 2rm \end{array} \right. \\ & \text{or} \\ & \left\{ \begin{array}{l} |\mu - \nu| = 2sm \\ \text{and} \\ \mu + \nu \neq 2rn \end{array} \right. \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

and also

$$\sum_{j=0}^{m}{}'' T_\mu(\eta_j^{(m)})\, T_\nu(\eta_j^{(m)}) = \begin{cases} m, & \begin{cases} \mu+\nu = 2pm \\ \text{and} \\ |\mu-\nu| = 2qm \end{cases} \\ \dfrac{m}{2}, & \begin{cases} \mu+\nu = 2pm \\ \text{and} \\ |\mu-\nu| \neq 2qm \end{cases} \\ & \text{or} \\ & \begin{cases} \mu+\nu \neq 2pm \\ \text{and} \\ |\mu-\nu| = 2qm \end{cases} \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

where the double stroke on a summation sign means that the first and last terms are to be halved. Equations (18) and (19) will play the role of (3) in this section.

2.1.

We begin by considering the zeros of the Chebyshev polynomials. Given $f \in A(C_\rho)$ we put

$$a_k^{(m)} = \frac{2}{m} \sum_{i=1}^{m} f(\xi_i^{(m)})\, T_k(\xi_i^{(m)}), \qquad k = 0, 1, 2, \ldots, \tag{20}$$

i.e., $a_k^{(m)}$ is the result of approximating A_k (given by (16)) by the appropriate Gaussian quadrature formula. Upon substituting (15) in (20) and using (18) we obtain, for $k < m$,

$$a_k^{(m)} = A_k + \sum_{j=1}^{\infty} (-1)^j \,(A_{2jm-k} + A_{2jm+k}), \tag{21}$$

and observe that (21) remains valid for $k = m$. We now put

$$u_{n,m}(f; z) = \sum_{k=0}^{n}{}' a_k^{(m)} T_k(z), \qquad n \leqslant m. \tag{22}$$

Let $L_k(f, T; z)$ be the polynomial of degree at most k which satisfies

$$L_k(f, T; \xi_i^{(k+1)}) = f(\xi_i^{(k+1)}), \qquad i = 1, \ldots, k+1,$$

for each $k = 0, 1, 2, \ldots$. Then it is known (cf. Rivlin [2]) that

$$u_{n,m}(f; z) = s_n(L_{m-1}(f, T); z), \qquad n < m \tag{23}$$

(and $u_{n,n}(f; z) = L_{n-1}(f, T; z)$, incidentally). Moreover, if $m > n$, then $u_{n,m}(f; z)$ is the least-squares approximation of degree n to $f(z)$ on $\{\xi_1^{(n)},..., \xi_m^{(m)}\}$. For,

$$\begin{aligned} u_{n,m}(f; z) &= u_{m-1,m}(f; z) - \sum_{k=n+1}^{m-1} a_k^{(m)} T_k(z) \\ &= L_{m-1}(f, T; z) - \sum_{k=n+1}^{m-1} a_k^{(m)} T_k(z) \end{aligned}$$

in view of (22) and (23). Thus, for $v = 0, 1,..., n$,

$$\sum_{j=1}^{m} [f(\xi_j^{(m)}) - u_{n,m}(f; \xi_j^{(m)})]\, T_v(\xi_j^{(m)}) = \sum_{\mu=n+1}^{m-1} a_\mu^{(m)} \left(\sum_{j=1}^{m} T_\mu(\xi_j^{(m)})\, T_v(\xi_j^{(m)}) \right) = 0$$

in view of (18), proving our assertion.

We are now in a position to discuss Walsh equiconvergence in the present setting. We have, for $m > n$, according to (21),

$$\begin{aligned} u_{n,m}(f; z) - s_n(f; z) &= \sum_{k=0}^{n}{}' (a_k^{(m)} - A_k)\, T_k(z) \\ &= \sum_{k=0}^{n}{}' \left(\sum_{j=1}^{\infty} (-1)^j (A_{2jm-k} + A_{2jm+k}) \right) T_k(z). \end{aligned} \tag{24}$$

We begin by considering the case $m = n + 1$, i.e., the difference (24) is $L_n(f, T; z) - s_n(f; z)$. Recall that $T_k(z) = (w^k + w^{-k})/2$, put $z = z_0 = [(\rho + \rho^{-1})/2] \in C_\rho$, and let

$$f(z) = f_0(z) = \frac{1 - \lambda z}{1 + \lambda^2 - 2\lambda z} = \sum_{k=0}^{\infty} \lambda^k T_k(z), \tag{25}$$

where $0 < \lambda = \rho^{-1} < 1$, in (24). Note that $f_0 \in A(C_\rho)$. Then

$$\begin{aligned} |u_{n,n+1}(f_0; z_0) - s_n(f_0; z_0)| &= \frac{\lambda^{2(n+1)}}{2(1 + \lambda^{2(n+1)})} \sum_{k=0}^{n}{}' (\lambda^k + \lambda^{-k})^2 \\ &> \frac{\lambda^{2(n+1)}}{4} \frac{\lambda^{-2n}}{2} = \frac{\lambda^2}{8}. \end{aligned}$$

Since $u_{n,n+1}(f_0; z_0) = L_n(f_0, T; z_0)$, we have arrived at the conclusion that a Walsh-type theorem in the present context does not hold for interpolation in the zeros of the Chebyshev polynomials.

To obtain some positive results, we return to (24) and put $m = nq + c$, where $q \geqslant 2$ is a fixed positive integer and c is any fixed integer. Equation (24) yields

$$|u_{n,m}(f; z) - s_n(f; z)| \leqslant \sum_{k=0}^{n}{}' \left(\sum_{j=1}^{\infty} (|A_{2jm-k}| + |A_{2jm+k}|)\right) \left(\frac{\tau^k + \tau^{-k}}{2}\right) \quad (26)$$

if z is inside or on C_τ, where we suppose $\tau \geqslant \rho$. Since the assumption that $f \in A(C_\rho)$ implies that as $k \to \infty$ $\overline{\lim} |A_k|^{1/k} = \rho^{-1}$ we see that given $\varepsilon > 0$ there exist $k_0(\varepsilon)$ such that for $k > k_0$

$$|A_k| < \left(\frac{1}{\rho}(1 + \varepsilon)\right)^k. \quad (27)$$

If we now put $r = ((1 + \varepsilon)/\rho) < 1$, use (27) in (26) and sum the resulting geometric progressions, we obtain, for n sufficiently large,

$$\begin{aligned} &|u_{n,m}(f; z) - s_n(f; z)| \\ &\leqslant \frac{1}{2} \frac{r^{2m}}{1 - r^{2m}} \sum_{k=0}^{n} \left[(\tau r)^k + \left(\frac{\tau}{r}\right)^k + \left(\frac{r}{\tau}\right)^k + \left(\frac{1}{r\tau}\right)^k\right] \\ &\leqslant 2 \frac{r^{2m}}{1 - r^{2m}} \frac{(\tau/r)^{n+1}}{(\tau/r) - 1} \leqslant M\tau^n r^{2m-n} = M\tau^n r^{(2q-1)n+c}, \end{aligned}$$

for some constant M. Thus we have proved

THEOREM 2. *If $f \in A(C_\rho)$ and q is an integer greater than 1 then*

$$\lim_{n \to \infty} (u_{n,m(n)}(f; z) - s_n(f; z)) = 0. \quad (28)$$

for z inside $C_{\rho^{2q-1}}$, the convergence being uniform and geometric inside and on C_τ for any $\tau < \rho^{2q-1}$, where $m(n) = nq + c$ and c is a fixed integer.

Remark 1. Equation (28) is sharp as can be seen by putting $f = f_0$ (as defined in (25)) and $z = [\rho^{2q-1} + \rho^{-(2q-1)}]/2$ in (24), and repeating, mutatis mutandis, the argument given in the case $m = n + 1$ above.

Remark 2. As $q \to \infty$, $u_{n,m}$ tends to s_n, which is the least-squares approximation of degree n to f on $[-1, 1]$ with the weight function $(1 - x^2)^{-1/2}$. This follows immediately from the fact of convergence of Gauss–Chebyshev quadratures.

2.2.

We turn next to the extrema of the Chebyshev polynomials. Given $f \in A(C_\rho)$, we put

$$b_k^{(m)} = \frac{2}{m} \sum_{i=0}^{m}{}'' f(\eta_i^{(m)})\, T_k(\eta_i^{(m)}), \qquad k = 0, 1, 2,...; \tag{29}$$

i.e., $b_k^{(m)}$ is the result of approximating A_k (given by (16)) by the Lobatto–Markov quadrature formula. Upon substituting (15) in (29) and using (19), we obtain, for $m \geqslant k$,

$$b_k^{(m)} = A_k + \sum_{j=1}^{\infty} (A_{2jm-k} + A_{2jm+k}). \tag{30}$$

If $m \geqslant n$, we put

$$v_{n,m}(f; z) = \sum_{k=0}^{n}{}'' b_k^{(m)} T_k(z). \tag{31}$$

Let $L_k(f, U; z)$ be the polynomial of degree at most k which satisfies

$$L_k(f, U; \eta_i^{(k)}) = f(\eta_i^{(k)}), \qquad i = 0,..., k,$$

for each $k = 0, 1, 2,...$. Then it is known (cf. Rivlin [2, Theorem 3.13]) that

$$v_{n,n}(f; z) = L_n(f, U; z). \tag{32}$$

Now

$$L_n(f, U; z) - s_n(f; z)$$
$$= \sum_{k=0}^{n-1}{}' \left(\sum_{j=1}^{\infty} (A_{2jn-k} + A_{2jn+k}) \right) T_k(z) + \left(\sum_{j=1}^{\infty} A_{(2j+1)n} \right) T_n(z),$$

and if we put $f = f_0$, as given by (25) and $z = z_0 = [(\rho + \rho^{-1})/2] \in C_\rho$ in this formula, we obtain

$$L_n(f_0, U; z_0) - s_n(f; z_0)$$
$$= \frac{\lambda^{2n}}{2(1 - \lambda^{2n})} \left[\sum_{k=0}^{n-1}{}' (\lambda^k + \lambda^{-k})^2 + \lambda^n(\lambda^n + \lambda^{-n}) \right] > \frac{\lambda^2}{2(1 - \lambda^2)}.$$

Thus we conclude that a Walsh-type theorem does not hold for interpolation in the extrema of the Chebyshev polynomial.

The polynomials $v_{n,m}(f; z)$ differ slightly from least-squares approximations to f. Namely, put

$$t_{n,m}(f; z) = \sum_{k=0}^{n}{}' b_k^{(m)} T_k(z) = v_{n,m}(f; z) + \tfrac{1}{2} b_n^{(m)} T_n(z).$$

We claim that, for $m \geqslant n+1$, $t_{n,m}(f; z)$ is the weighted least-squares approximation of degree n to $f(z)$ on $\{\eta_0^{(m)},..., \eta_m^{(m)}\}$, the weight 1 being associated with $\eta_i^{(m)}$, $0 < i < m$ and weight 1/2 with $\eta_0^{(m)}, \eta_m^{(m)}$. For, if $m \geqslant n+1$,

$$\begin{aligned} t_{n,m}(f; z) &= \sum_{k=0}^{m}{}'' \, b_k^{(m)} T_k(z) - \sum_{k=n+1}^{m}{}^{*} \, b_k^{(m)} T_k(z) \\ &= L_m(f, U; z) - \sum_{k=n+1}^{m}{}^{*} \, b_k^{(m)} T_k(z), \end{aligned} \tag{33}$$

where the asterisk on the summation sign means that the last term is to be halved. Thus, for $\nu = 0, 1,..., n$,

$$\begin{aligned} &\sum_{j=0}^{m}{}'' \, (f(\eta_j^{(m)}) - t_{n,m}(f; \eta_j^{(m)})) \, T_\nu(\eta_j^{(m)}) \\ &\qquad = \sum_{\mu=n+1}^{m}{}^{*} \, b_\mu^{(m)} \left(\sum_{j=0}^{m}{}'' \, T_\mu(\eta_j^{(m)}) \, T_\nu(\eta_j^{(m)}) \right) = 0, \end{aligned}$$

in view of (19), thus establishing our claim.

We observe next that in view of (30)

$$\begin{aligned} t_{n,m}(f; z) - s_n(f; z) &= \sum_{k=0}^{n}{}' \, (b_k^{(m)} - A_k) \, T_k(z) \\ &= \sum_{k=0}^{n}{}' \left(\sum_{j=1}^{\infty} (A_{2jm-k} + A_{2jm+k}) \right) T_k(z). \end{aligned} \tag{34}$$

Upon comparing (34) to (24) we see that if we replace $u_{n,m}(f; z)$ by $t_{n,m}(f; z)$ in Theorem 2 an equally valid result is obtained. But examining (34) and (24) also suggests that we should examine the average,

$$w_{n,m}(f; z) = \frac{u_{n,m}(f; x) + t_{n,m}(f; x)}{2}, \tag{35}$$

of the least-squares approximations of f on the Chebyshev zeros and extrema. For (34) and (24) imply that

$$w_{n,m}(f; z) - s_n(f; z) = \sum_{k=0}^{n}{}' \left(\sum_{j=1}^{\infty} (A_{4jm-k} + A_{4jm+k}) \right) T_k(z), \tag{36}$$

and the same argument that led to Theorem 2 now provides

THEOREM 3. *If $f \in A(C_\rho)$ and q is a positive integer, then*

$$\lim_{n \to \infty} (w_{n,m(n)}(f; z) - s_n(f; z)) = 0, \tag{37}$$

for z inside $C_{\rho^{4q-1}}$, the convergence being uniform and geometric inside and on C_τ for any $\tau < \rho^{4q-1}$, where $m(n) = nq + c$ and c is a fixed integer.

Remark 1. Equation (37) is sharp as can be seen by putting $f = f_0$ (as defined in (25)) and $z = [(\rho^{4q-1} + \rho^{-(4q-1)})/2]$ in (36).

Remark 2. The case $m(n) = n + 1$, i.e., $q = 1$, $c = 1$ is particularly interesting. For, in this case, according to (23), $u_{n,n+1}(f; z) = L_n(f, T; z)$, while $t_{n,n+1}(f; z)$ is the best uniform approximation to $f(z)$ by a polynomial of degree at most n on $\{\eta_0^{(n+W)},..., \eta_{(n+1)}^{(n+1)}\}$. This latter fact is easily seen since (33) yields

$$t_{n,n+1}(f; z) = L_{n+1}(f, U; z) - \tfrac{1}{2} b_{(n+1)}^{(n+1)} T_{n+1}(z),$$

so that for $j = 0,..., n + 1$,

$$f(\eta_j^{(n+1)}) - t_{n,n+1}(f; \eta_j^{(n+1)}) = (-1)^j \frac{b_{n+1}^{(n+1)}}{2},$$

and we have Chebyshev alternation. An argument, by now familiar, proceeding from (34) shows that $t_{n,n+1}(f; z)$ does not exhibit Walsh equiconvergence for $f = f_0$ and $z = z_0$. We have already seen that the same is true of $L_n(f, T; z)$. However, our theorem tells us that the average of interpolation in the Chebyshev zeros and best uniform approximation on the Chebyshev extrema does have the Walsh equiconvergence property within C_{ρ^3}.

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